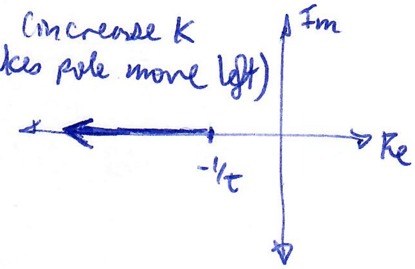
ME 4555 – Lecture 24 – Root locus

We saw that using proportional control caused the poles to move around. As a reminder:

$$G(s) = \frac{1}{\tau s + 1}, \quad C(s) = K \qquad \text{produces:}$$

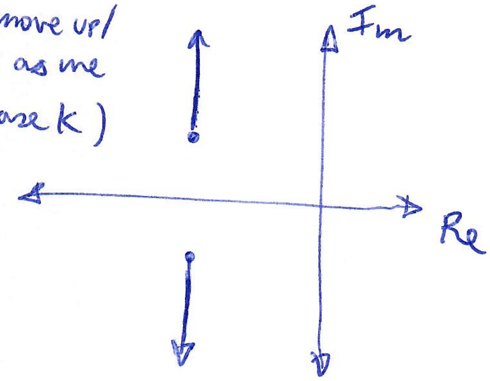
(increase K makes pole move left)



$$\frac{GC}{1+GC} = \frac{K}{\tau s + 1 + K}$$

Second example: $$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

(pole move up/down as we increase K)



$$C(s) = K$$

$$\frac{GC}{1+GC} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + (K+1)\omega_n^2}$$

We will now learn how to predict the way poles will move if we use proportional gain. This diagram of how poles move is called a <u>root locus</u>.

Suppose our plant is: $$G(s) = \frac{b(s)}{a(s)} = \frac{b_m s^m + \dots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0}$$

assume $n \geq m$ (physically realisable system)

the <u>zeros</u> of G ("open-loop zeros") are the roots of $b(s)$.
i.e. solutions to the equation $b(s) = 0$.

the <u>poles</u> of G ("open-loop poles") are the roots of $a(s)$.
i.e. solutions to the equation $a(s) = 0$.

The closed loop is (assuming proportional gain K):

$$\frac{KG(s)}{1+KG(s)} = \frac{\frac{Kb(s)}{a(s)}}{1+\frac{Kb(s)}{a(s)}} = \frac{Kb(s)}{a(s)+Kb(s)}$$

$\boxed{\text{Property 1 (zeros)}}$ The closed-loop zeros are roots of $b(s)$. This is the same as the open-loop zeros! Therefore, feedback does <u>not</u> change the location of the zeros.

$\boxed{\text{Property 2 (poles)}}$ The denominator is a polynomial $a(s)+Kb(s)$ that has the same degree, n, as $a(s)$ (since $n \geq m$). So the closed-loop poles occur in conjugate pairs, i.e. the root locus is symmetric about the real axis. There are exactly n poles.

$\boxed{\text{Property 3 (limits)}}$ When $K \approx 0$, denominator is $a(s)$. So closed-loop poles are the same as open-loop poles. As $K \to \infty$, denominator looks more like $b(s)$. Closed-loop poles become open-loop zeros. Each pole follows a path ("branch"). Separate poles have separate branches, starting at ϕ poles and ending at ϕ zeros. Each pole matches to a different zero. Excess poles go to ∞.

<u>Intuition</u>: the poles "repel" each other and are "attracted" to the zeros (which are fixed).

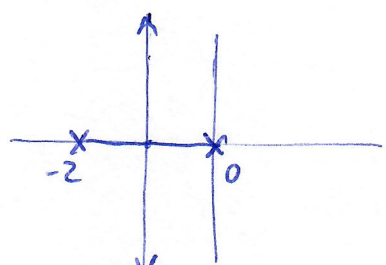
Property 4 (asymptotes) Since m poles go to the m zeros, the remaining $n-m$ poles go to ∞. They go in straight lines asymptotically. If poles are at $p_1, \dots, p_n$ and zeros are at $z_1, \dots, z_m$, then the centroid is at $\sigma_A = \dfrac{\Sigma p_i - \Sigma z_i}{n-m}$.

The angles made by the asymptotes are $\phi_A = \dfrac{(1 \pm 2k)\pi}{n-m}$, $k = 0, 1, \dots$

Angles do not depend on pole/zero locations, only their number.

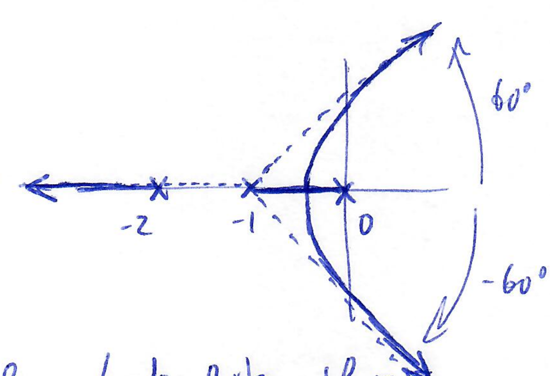
<u>Ex</u>: $\dfrac{1}{s(s+2)}$. poles at $\{0, -2\}$. $n=2$, $m=0$. Centroid: $\sigma_A = \frac{-2}{2} = -1$. angles: $\phi_A = \{\pm \frac{\pi}{2}\}$.

Root locus:

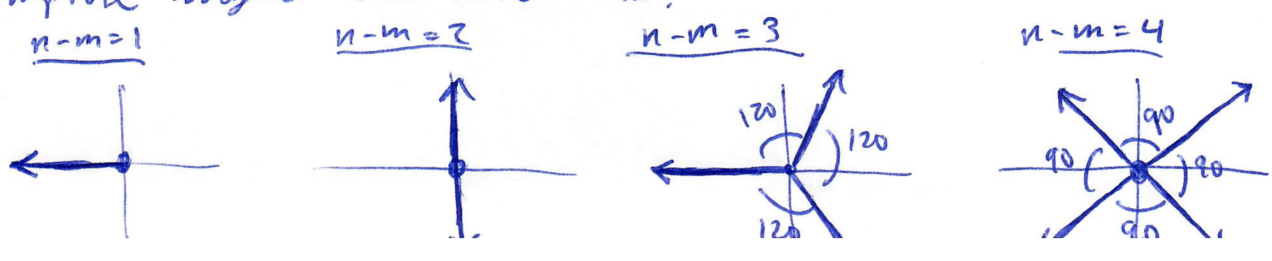


<u>Ex</u>: $\dfrac{1}{s(s+1)(s+2)}$ poles at $\{0, -1, -2\}$ $n=3$, $m=0$. Centroid: $\sigma_A = \frac{-3}{3} = -1$. angles: $\phi_A = \{\pm \frac{\pi}{3}, \pi\}$.

Root locus:



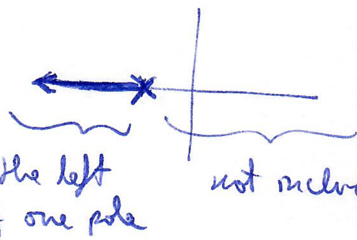
Asymptote angles look like this:



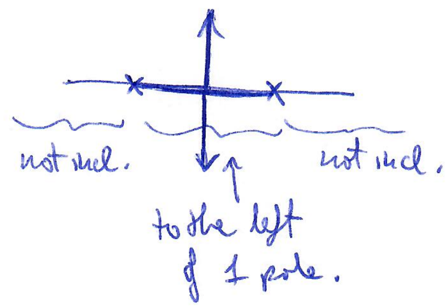
etc...

Property 5 (real axis) the parts of the root locus including the real axis are precisely the segments to the left of an odd number of real poles or zeros. If there are no real poles or zeros, the root locus does not touch the real axis.

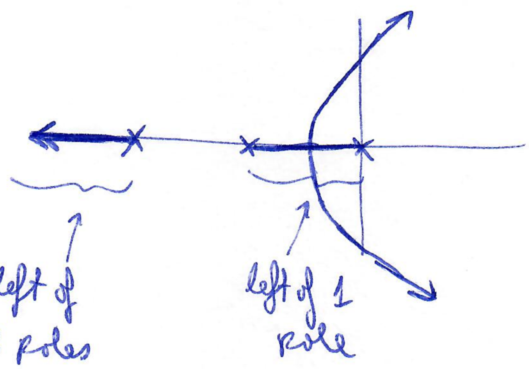
Ex: $\frac{1}{s+1}$



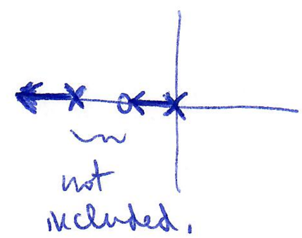
Ex: $\frac{1}{(s+1)(s-1)}$



Ex: $\frac{1}{s(s+1)(s+2)}$

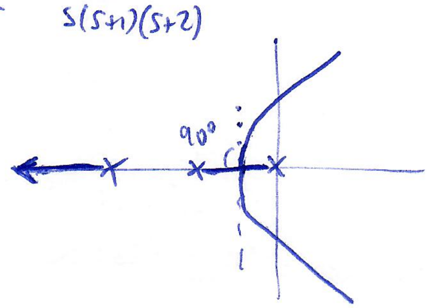


Ex: $\frac{s+1}{s(s+2)}$

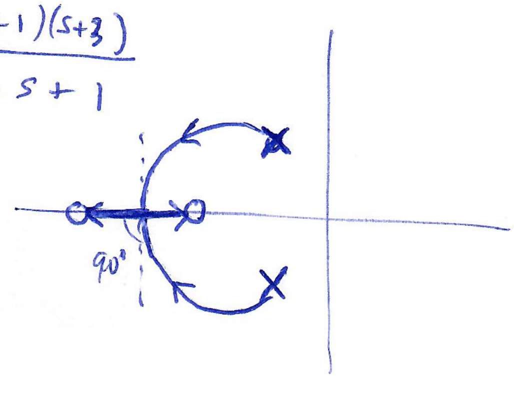


Property 6 (break in/out) Any time the root locus breaks away from the real axis ("break-out") or arrives into the real axis ("break-in"), it does so at an angle of $\pm 90°$.

Ex $\frac{1}{s(s+1)(s+2)}$

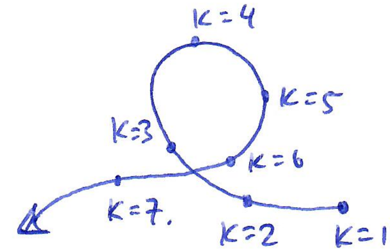


Ex $\frac{(s+1)(s+3)}{s^2+s+1}$



$\boxed{\text{Property 7 (no crossing)}}$ A root locus cannot cross itself.

For example, we will <u>never</u> have:



These properties are sufficient to sketch most root locus plots.

Good order to follow (sketching procedure).

1) identify and label ⊗ poles and ⊘ zeros. (Prop 1-3).
2) identify parts of real axis that belong to locus (Prop. 5)
3) identify centroid + asymptotes. (Prop. 4),
4) sketch! (Prop. 3, 6, 7).

It's also possible to figure out:

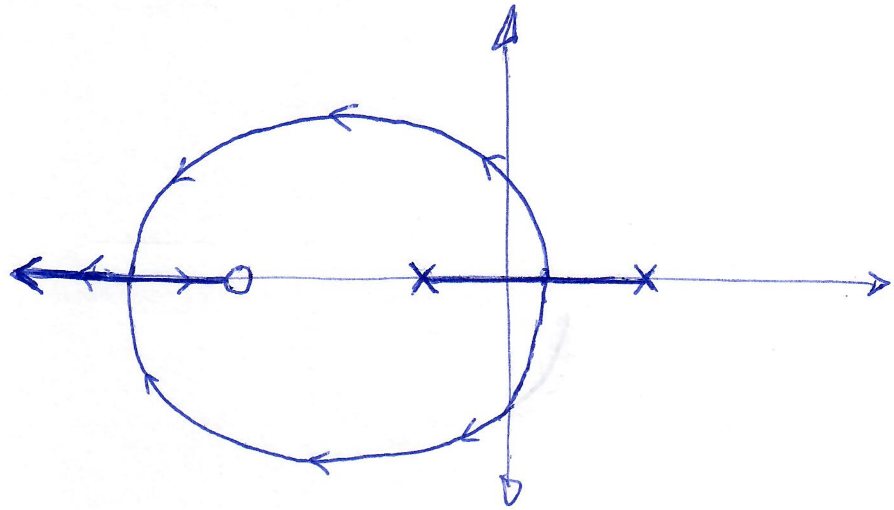
- exact break in/out locations by solving $\frac{d}{ds}\left(G(s)\right) = 0$ for s.
- where locus crosses imaginary axis (solve $a(j\omega) + kb(j\omega) = 0$)
- departure/arrival angles for complex poles/zeros.
- negative locus $(k < 0)$ (flip prop. 4 & 5 about $j\omega$-axis).

Examples

$$\frac{s+3}{s^2-s-2} = \frac{(s+3)}{(s+1)(s-2)}$$

poles = $\{-1, 2\}$

zeros = $\{-3\}$.

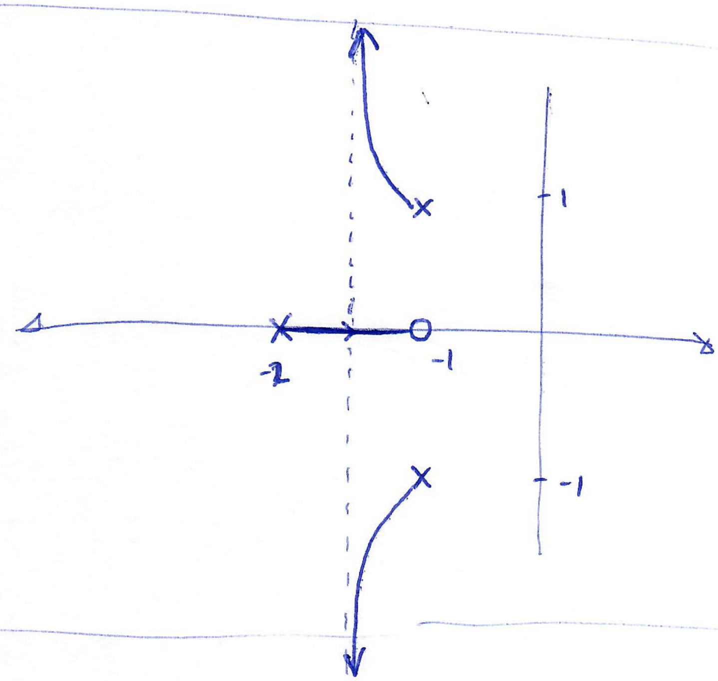


$$\frac{s+1}{s^3+4s^2+6s+4} = \frac{(s+1)}{(s+2)(s^2+2s+2)}$$

poles : $\{-2, -1 \pm i\}$

zeros : $\{-1\}$

$$\sigma_A = \frac{(-2)+(-1+i)+(-1-i)-(-1)}{2} = -\frac{3}{2}$$



similar arrangements can lead to branch "jumps" (same # poles and zeros, so similar asymptotes)